

*This paper is dedicated to Professor Vladimir Gutlyanskii on the occasion of his 75-th anniversary.*

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## ON QUASILINEARLY SUBHARMONIC FUNCTIONS

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**ABSTRACT.** We recall the definition of quasilinearly subharmonic functions, point out that this function class includes, among others, subharmonic functions, quasisubharmonic functions, nearly subharmonic functions and essentially almost subharmonic functions. It is shown that the sum of two quasilinearly subharmonic functions may not be quasilinearly subharmonic. Moreover, we characterize the harmonicity via quasilinearly subharmonicity.

### 1. SUBHARMONIC FUNCTIONS AND NEARLY SUBHARMONIC FUNCTIONS.

Denote by  $\mathbb{R}^N$  the  $N$ -dimensional Euclidean space. If  $x \in \mathbb{R}^N$ , then the open ball centered at  $x$  with radius  $r > 0$  will be denoted by  $B^N(x, r)$  and we will write  $\overline{B^N(x, r)}$  for the closure of this ball.

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*2010 Mathematical Subject Classification.* Primary 31B05, 31C05; Secondary 31C45.

*Key words and phrases.* Subharmonic, quasilinearly subharmonic, nearly subharmonic.

The publication is based on the research provided by the grant support of the State Fund For Fundamental Research (project N 20570). The first author was also partially supported by Project 15-1bb\19 “Metric Spaces, Harmonic Analysis of Functions and Operators and Singular and Nonclassic Problems for Differential Equations” (Donetsk National University, Vinnitsia, Ukraine).

Let  $D$  be a domain in  $\mathbb{R}^N$ ,  $N \geq 2$ . An upper semicontinuous function  $u : D \rightarrow [-\infty, +\infty)$  is *subharmonic* if the inequality

$$u(x) \leq \frac{1}{\nu_N r^N} \int_{B^N(x,r)} u(y) dm_N(y)$$

holds for all  $\overline{B^N(x,r)} \subset D$ , where  $\nu_N$  is the volume of the unit ball in  $\mathbb{R}^N$ .

The function  $u \equiv -\infty$  is considered subharmonic. A function  $u$  defined on an open set  $\Omega \subseteq \mathbb{R}^N$  is subharmonic if the restriction of  $u$  to arbitrary connected component of  $\Omega$  is subharmonic.

**Definition 1.** A function  $u : D \rightarrow [-\infty, +\infty)$  is *nearly subharmonic*, if  $u$  is Lebesgue measurable,  $u^+ \in \mathcal{L}_{\text{loc}}^1(D)$  and

$$(1) \quad u(x) \leq \frac{1}{\nu_N r^N} \int_{B^N(x,r)} u(y) dm_N(y)$$

holds for all  $\overline{B^N(x,r)} \subset D$ .

Observe that our definition is slightly nonstandard because in the standard definition of nearly subharmonic functions one uses the stronger assumption  $u \in \mathcal{L}_{\text{loc}}^1(D)$ , see e.g. [16], p. 14.

The following lemma is an analog of Proposition 2.2 (vii) from [36], p. 55, and Proposition 1.5.2 (vii) from [39], p. e2615.

**Lemma 1.** Let  $D$  be a domain in  $\mathbb{R}^N$ ,  $N \geq 2$ , and let  $u : D \rightarrow [-\infty, +\infty)$  be nearly subharmonic in the sense of Definition 1. Then either  $u \in \mathcal{L}_{\text{loc}}^1(D)$  or the equality  $u(x) = -\infty$  holds for every  $x \in D$ .

*Proof.* Suppose  $u \notin \mathcal{L}_{\text{loc}}^1(D)$ . Then there is a compact set  $K \subset D$  such that

$$(2) \quad \int_K u(y) dm_N(y) = -\infty.$$

Since  $K$  is compact and  $D$  is open, we have

$$\text{dist}(K, \partial D) = \inf_{x \in K, y \in \partial D} |x - y| > 0.$$

Let  $\varepsilon$  be a positive real number satisfying the inequality

$$(3) \quad 3\varepsilon < \text{dist}(K, \partial D).$$

We can find a finite set of balls  $B^N(x_1, \varepsilon), \dots, B^N(x_m, \varepsilon)$  such that  $x_i \in K$  for every  $i \in \{1, \dots, m\}$  and

$$K \subseteq \bigcup_{i=1}^m B^N(x_i, \varepsilon) \subseteq D.$$

These inclusions and (2) imply

$$(4) \quad \int_{B^N(x_{i_0}, \varepsilon)} u(y) dm_N(y) = -\infty$$

for some  $i_0 \in \{1, \dots, m\}$ . It follows from (3) and  $x_{i_0} \in K$  that

$$B^N(x_{i_0}, \varepsilon) \subset B^N(x, 2\varepsilon) \subseteq D$$

holds for every  $x \in B^N(x_{i_0}, \varepsilon)$ . Using (4) we obtain

$$\int_{B^N(x, 2\varepsilon)} u(y) dm_N(y) = -\infty$$

for every  $x \in B^N(x_{i_0}, \varepsilon)$ . Since  $u$  is nearly subharmonic, it follows that

$$-\infty \leq u(x) \leq \frac{1}{\nu_N(2\varepsilon)^N} \int_{B^N(x, 2\varepsilon)} u(y) dm_N(y) = -\infty,$$

i.e.,  $u(x) = -\infty$  for every  $x \in B^N(x_{i_0}, \varepsilon)$ . Write

$$A = \{x \in D : u(x) = -\infty\}.$$

Since  $B^N(x_{i_0}, \varepsilon) \subseteq A$ , the interior of  $A$  is non-void,  $\text{Int}(A) \neq \emptyset$ . To complete the proof, it is sufficient to show that  $\text{Int}(A) = D$ . If the last equality does not hold, then there is a point  $y^* \in D \cap \partial \text{Int}(A)$ . Let  $0 < \delta^* < \frac{1}{2} \text{dist}(y^*, \partial D)$ . Then for every  $y \in B^N(y^*, \delta^*)$  we have

$$D \supseteq B^N(y, 2\delta^*) \text{ and } B^N(y, 2\delta^*) \cap \text{Int}(A) \neq \emptyset.$$

Consequently  $u(y) = -\infty$  holds for every  $y \in B^N(y^*, \delta^*)$ . Thus  $y^* \in \text{Int}(A)$ , contrary to  $y^* \in \partial \text{Int}(A)$ .  $\square$

The following proposition is well known under the additional condition  $u \in \mathcal{L}_{\text{loc}}^1(D)$ .

**Proposition 1.** *Let  $D$  be a domain in  $\mathbb{R}^N$ ,  $N \geq 2$  and let  $u : D \rightarrow [-\infty, +\infty)$  be Lebesgue measurable. Then  $u$  is nearly subharmonic in  $D$  if and only if there exists a subharmonic in  $D$  function  $u^*$  such that  $u^*(x) \geq u(x)$  for all  $x \in D$  and  $u^*(x) = u(x)$  holds Lebesgue almost everywhere.*

*Proof.* If  $u(x) \equiv -\infty$  then, the proposition is evident. In the opposite case by Lemma 1 we have  $u \in \mathcal{L}_{\text{loc}}^1(D)$ , and it is a reformulation of Theorem 1 from [16], p. 14.  $\square$

**Remark 1.** *In particular, if  $u$  is nearly subharmonic, then we can take  $u^*$  as the lowest upper semicontinuous majorant of  $u$ :*

$$u^*(x) = \limsup_{x' \rightarrow x} u(x').$$

Observe also that the *almost subharmonic functions*, by Szpilrajn [47], are included in Definition 1 in the following sense. Let  $u : D \rightarrow [-\infty, +\infty)$  be almost subharmonic, that is,  $u \in \mathcal{L}_{\text{loc}}^1(D)$  and inequality (1) is satisfied for Lebesgue almost every  $x \in D$  with all  $\overline{B^N(x, r)} \subset D$ . Let

$$D_1 := \{x \in D : u(x) \leq \frac{1}{\nu_N r^N} \int_{B^N(x, r)} u(y) dm_N(y) \text{ for all } \overline{B^N(x, r)} \subset D\}.$$

Define  $\tilde{u} : D \rightarrow [-\infty, +\infty)$  as

$$\tilde{u}(x) := \begin{cases} u(x), & \text{when } x \in D_1, \\ -\infty, & \text{when } x \in D \setminus D_1. \end{cases}$$

By assumption  $m_N(D \setminus D_1) = 0$ , it is easy to see that  $\tilde{u}$  is nearly subharmonic in  $D$ .

In the connection with almost subharmonic functions see also [3] and [26], p. 20, and [20], p. 238. Lieb and Loss even call the almost subharmonic functions briefly subharmonic functions.

## 2. QUASINEARLY SUBHARMONIC FUNCTIONS

Let  $D$  be a domain in  $\mathbb{R}^N$ ,  $N \geq 2$ . It is an important fact that if  $u : D \rightarrow [0, +\infty)$  is subharmonic and  $p > 0$ , then there exists a constant  $K = K(N, p) > 0$  such that the inequality

$$(5) \quad u(x)^p \leq \frac{K}{\nu_N r^N} \int_{B^N(x, r)} u(y)^p dm_N(y)$$

holds for all  $\overline{B^N(x, r)} \subset D$ . In the case of  $p = 1$  and  $K = 1$ , inequality (5) is just the familiar mean value inequality for (nonnegative) subharmonic functions. The case  $p > 1$  follows from the case  $p = 1$  with the aid of Jensen's inequality. The case  $0 < p < 1$  has been given in Fefferman and Stein [12], Lemma 2, p. 172 and in [19], Theorem 1, p. 529, where, however, only absolute values of harmonic functions were considered. The proofs in [12] and in [19] are somewhat long. See also [13], Lemma 3.7, p. 116, and [2], (1.5), p. 210. In [27], Lemma, p. 69, it was pointed out that the proof in [12] includes the case of nonnegative subharmonic functions, too. See also [45], p. 271, [46], p. 114, [15], Lemma 1, p. 113, [40], Lemma 3, p. 305, [41], p. 794, [42], [43], Lemma 1, p. 363, [44], Lemma 2.1, p. 7, [6], Theorem A, p. 413, and [1], p. 132. Observe that a possibility for an essentially different proof was pointed out already in [48], pp. 188-190. Later other different proofs were given in [23], p. 18 and Theorem 1, p. 19 (see also [24], Theorem A, p. 15), [28], pp. 233-234, [29], p. 188. The results in [23], [28] and [29] hold in fact for more general function classes than just for nonnegative subharmonic functions. Compare also [4], [7], p. 430, and [8], p. 485.

Inequality (5) has many applications. Among others, it has been applied to the weighted boundary behavior of subharmonic functions and to the nonintegrability of subharmonic or superharmonic functions.

It is therefore relevant to find a generalization of results related to inequality (5). We will do this in the following way.

Let  $D$  be a domain in  $\mathbb{R}^N$ ,  $N \geq 2$ . For every  $u : D \rightarrow [-\infty, +\infty)$  and  $M \geq 0$  we write  $u_M := \max\{u, -M\} + M$ .

**Definition 2.** Let  $K \in [1, +\infty)$ . A Lebesgue measurable function  $u : D \rightarrow [-\infty, +\infty)$  is  $K$ -quasilinearly subharmonic, if  $u^+ \in \mathcal{L}_{\text{loc}}^1(D)$  and the inequality

$$u_M(x) \leq \frac{K}{\nu_N r^N} \int_{B^N(x,r)} u_M(y) dm_N(y)$$

holds for all  $M \geq 0$  and  $\overline{B^N(x,r)} \subset D$ . A function  $u : D \rightarrow [-\infty, +\infty)$  is quasilinearly subharmonic, if  $u$  is  $K$ -quasilinearly subharmonic for some  $K$ .

In addition to the above defined class of quasilinearly subharmonic functions, we will consider their proper subclass.

**Definition 3.** A Lebesgue measurable function  $u : D \rightarrow [-\infty, +\infty)$  is  $K$ -quasilinearly subharmonic n.s. (in the narrow sense), if  $u^+ \in \mathcal{L}_{\text{loc}}^1(D)$  and if there is a constant  $K = K(N, u, D) \geq 1$  such that the inequality

$$u(x) \leq \frac{K}{\nu_N r^N} \int_{B^N(x,r)} u(y) dm_N(y)$$

holds for all  $\overline{B^N(x,r)} \subset D$ . A function  $u : D \rightarrow [-\infty, +\infty)$  is quasilinearly subharmonic n.s., if  $u$  is  $K$ -quasilinearly subharmonic n.s. in  $D$  for some  $K$ .

For a function  $u$  is defined on an open set  $\Omega \subseteq \mathbb{R}^n$ , the quasilinearly subharmonicity (quasilinearly subharmonicity n.s.) of  $u$  means that the restriction of  $u$  to arbitrary connected component of  $\Omega$  is quasilinearly subharmonic (quasilinearly subharmonic n.s.).

Observe that if  $u : D \rightarrow [0, +\infty)$  is subharmonic and  $p > 0$ , then  $u^p$  is quasilinearly subharmonic n.s. and thus also quasilinearly subharmonic, see statement (1) and statement (4) of Proposition 2 below and also [11].

More generally, the class of quasilinearly subharmonic functions includes, among others the subharmonic and nearly subharmonic functions and also the quasisubharmonic functions (for the definition of this see [36] and [16]), also functions satisfying certain natural growth conditions, especially certain eigenfunctions, polyharmonic functions, subsolutions of certain general elliptic equations.

Let  $D$  be a domain in  $\mathbb{R}^N$ ,  $N \geq 2$ . Recall that a continuous function  $u : D \rightarrow [0, \infty)$  is said to be a Harnack function if there are  $\lambda \in (0, 1)$  and  $C_\lambda \in [1, \infty)$  such that the following Harnack inequality

$$\max_{z \in B^N(x, \lambda r)} u(z) \leq C_\lambda \min_{z \in B^N(x, \lambda r)} u(z)$$

holds whenever  $B^N(x, r) \subseteq D$ . See [49], p. 259. Every Harnack function is quasilinearly subharmonic. This implies the quasilinearly subharmonicity of nonnegative harmonic functions as well as nonnegative solutions of some elliptic equations. In particular, the partial differential equations associated with quasiregular mappings belong to this family of elliptic equations, see Vuorinen [49] and the above references.

Observe that already Domar [7] has pointed out the relevance of the class of (nonnegative) quasilinearly subharmonic functions. For, at least partly, more general function class, see [8].

We list below four simple examples of quasilinearly subharmonic functions.

**Example 1.** Let  $D$  be a domain in  $\mathbb{R}^N$ ,  $N \geq 2$ . Any Lebesgue measurable function  $u : D \rightarrow [m, M]$ , where  $0 < m \leq M < +\infty$ , is quasilinearly subharmonic n.s. and, because of Proposition 2 (see below), also quasilinearly subharmonic. If  $u$  is moreover continuous, then  $u$  is a Harnack function.

**Example 2.** The function  $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,

$$u(x, y) := \begin{cases} -1, & \text{when } y < 0 \\ 1, & \text{when } y \geq 0, \end{cases}$$

is 2-quasilinearly subharmonic, but not quasilinearly subharmonic n.s..

**Example 3.** Let  $D = (0, 2) \times (0, 1) \subset \mathbb{R}^2$  and let  $c < 0$  be arbitrary. Let  $E \subset D$  be a Borel set of zero Lebesgue measure. Let  $u : D \rightarrow [-\infty, +\infty)$ ,

$$u(x, y) := \begin{cases} c, & \text{when } (x, y) \in E, \\ 1, & \text{when } (x, y) \in D \setminus E \text{ and } 0 < x < 1, \\ 2, & \text{when } (x, y) \in D \setminus E \text{ and } 1 \leq x < 2. \end{cases}$$

The function  $u$  attains both negative and positive values, is 2-quasilinearly subharmonic n.s., but not nearly subharmonic.

**Example 4.** Let  $D$  be a domain in  $\mathbb{R}^N$ ,  $N \geq 2$ , and let  $u : D \rightarrow [-\infty, +\infty)$  be a quasilinearly subharmonic function (resp. quasilinearly subharmonic n.s.). Let  $E \subset D$  be a Borel set of zero Lebesgue measure. Let  $v : D \rightarrow [-\infty, +\infty)$ ,

$$v(x) := \begin{cases} -\infty, & \text{when } x \in E, \\ u(x), & \text{when } x \in D \setminus E. \end{cases}$$

The function  $v$  is quasilinearly subharmonic (resp. quasilinearly subharmonic n.s.).

The term quasilinearly subharmonic function was first introduced in [30]. Quasilinearly subharmonic functions (sometimes with a different terminology), or, essentially, perhaps just functions satisfying a certain generalized mean value inequality, have previously been considered, or used, in addition to the above listed references at least in [22], [30], [31], [32], [34], [35], [5], [17], [37], [38], [9], [10], [18] and [21].

### 3. BASIC PROPERTIES OF QUASILINEARLY SUBHARMONIC FUNCTIONS

Recall that a function  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  satisfies a  $\Delta_2$ -condition, if there is a constant  $C = C(\varphi) \geq 1$  such that  $\varphi(2t) \leq C \varphi(t)$  for all  $t \in [0, +\infty)$ .

**Definition 4.** A function  $\psi : [0, +\infty) \rightarrow [0, +\infty)$  is permissible, if there exist an increasing (strictly or not), convex function  $\psi_1 : [0, +\infty) \rightarrow [0, +\infty)$  and a strictly increasing surjection  $\psi_2 : [0, +\infty) \rightarrow [0, +\infty)$  such that  $\psi = \psi_2 \circ \psi_1$  and the following conditions hold:

- (a)  $\psi_1$  satisfies the  $\Delta_2$ -condition,
- (b)  $\psi_2^{-1}$  satisfies the  $\Delta_2$ -condition,
- (c) the function  $t \mapsto \frac{\psi_2(t)}{t}$  is quasi-decreasing, i.e. there is a constant  $C = C(\psi_2) > 0$  such that

$$\frac{\psi_2(s)}{s} \geq C \frac{\psi_2(t)}{t}$$

whenever  $0 < s \leq t$ .

Permissible functions are necessarily continuous.

Examples of permissible functions are:  $\psi_1(t) = t^p$ ,  $p > 0$ , and  $\psi_2(t) = c t^{p\alpha} [\log(\delta + t^{p\gamma})]^\beta$ ,  $c > 0$ ,  $0 < \alpha < 1$ ,  $\delta \geq 1$ ,  $\beta, \gamma \in \mathbb{R}$  such that  $0 < \alpha + \beta\gamma < 1$ , and  $p \geq 1$ . And also functions of the form  $\psi_3 = \phi \circ \varphi$ , where  $\phi : [0, +\infty) \rightarrow [0, +\infty)$  is a concave surjection whose inverse  $\phi^{-1}$  satisfies the  $\Delta_2$ -condition and  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  is an increasing, convex function satisfying the  $\Delta_2$ -condition.

It is interesting to note the following fact, see [25], Lemma 1 and Remark 1, p. 93:

Let  $\psi : [0, +\infty) \rightarrow [0, +\infty)$  be a permissible function. Then

- (1) there are a number  $p > 0$  and a convex function  $M : [0, +\infty) \rightarrow [0, +\infty)$  satisfying the  $\Delta_2$ -condition such that  $\psi(t) \asymp g(t^p)$ , that is, there exist constants  $C_1 > 0$  and  $C_2 > 0$  such that

$$C_1 \leq \frac{\psi(t)}{g(t^p)} \leq C_2$$

for all  $t > 0$ ;

- (2) there are a number  $p > 0$  and a convex function  $\vartheta : [0, +\infty) \rightarrow [0, +\infty)$  satisfying the  $\Delta_2$ -condition such that  $\psi(t) \asymp \vartheta(t)^p$ .

Next we list certain basic properties of quasinearly subharmonic functions, see [36], Proposition 2.1 and Proposition 2.2 and [39], Proposition 1.5.1.

**Proposition 2.** Let  $D$  be a domain in  $\mathbb{R}^N$ ,  $N \geq 2$ .

- (1) If  $u : D \rightarrow [0, +\infty)$  is Lebesgue measurable and  $u^+ \in \mathcal{L}_{loc}^1(D)$ , then  $u$  is  $K$ -quasinearly subharmonic if and only if  $u$  is  $K$ -quasinearly subharmonic n.s., that is, if

$$u(x) \leq \frac{K}{\nu_N r^N} \int_{B^N(x,r)} u(y) dm_N(y)$$

holds for all  $\overline{B^N(x,r)} \subset D$ .

- (2) If  $u : D \rightarrow [-\infty, +\infty)$  is  $K$ -quasilinearly subharmonic n.s., then  $u$  is  $K$ -quasilinearly subharmonic in  $D$ , but not necessarily conversely.
- (3) A function  $u : D \rightarrow [-\infty, +\infty)$  is 1-quasilinearly subharmonic if and only if it is nearly subharmonic, that is, 1-quasilinearly subharmonic n.s.
- (4) If  $u : D \rightarrow [0, +\infty)$  is quasilinearly subharmonic and  $\psi : [0, +\infty) \rightarrow [0, +\infty)$  is permissible, then  $\psi \circ u$  is quasilinearly subharmonic in  $D$ . Especially, if  $h : D \rightarrow \mathbb{R}$  is harmonic and  $p > 0$ , then  $|h|^p$  is quasilinearly subharmonic.
- (5) The Harnack functions are quasilinearly subharmonic.

*Proof.* We leave statements (1), (2) and (5) to the reader. For the proof of statement (4), see [33], Lemma 2.1, p. 32. To prove statement (3) suppose that  $u$  is nearly subharmonic in  $D$ . Then clearly  $u_M$  is nearly subharmonic for all  $M \geq 0$ , and thus for every  $\overline{B^N}(x, r) \subset D$ , one has

$$u_M(x) \leq \frac{1}{\nu_N r^N} \int_{B^N(x, r)} u_M(y) dm_N(y).$$

Hence  $u$  is 1-quasilinearly subharmonic.

On the other hand, if  $u$  is 1-quasilinearly subharmonic in  $D$ , then one sees at once, with the aid of the Lebesgue Monotone Convergence Theorem, that  $u$  is nearly subharmonic in  $D$ .  $\square$

**Proposition 3.** Let  $D$  be a domain in  $\mathbb{R}^N$ ,  $N \geq 2$ .

- (1) If  $u : D \rightarrow [-\infty, +\infty)$  is  $K_1$ -quasilinearly subharmonic and  $K_2 \geq K_1$ , then  $u$  is  $K_2$ -quasilinearly subharmonic.
- (2) If  $u_1 : D \rightarrow [-\infty, +\infty)$  and  $u_2 : D \rightarrow [-\infty, +\infty)$  are  $K$ -quasilinearly subharmonic n.s., then  $\lambda_1 u_1 + \lambda_2 u_2$  is  $K$ -quasilinearly subharmonic n.s. for all  $\lambda_1, \lambda_2 \geq 0$ .
- (3) If  $u : D \rightarrow [-\infty, +\infty)$  is quasilinearly subharmonic, then  $u$  is locally bounded above.
- (4) If  $u_j : D \rightarrow [-\infty, +\infty)$ ,  $j = 1, 2, \dots$ , are  $K$ -quasilinearly subharmonic (resp.  $K$ -quasilinearly subharmonic n.s.), and  $u_j \searrow u$  as  $j \rightarrow +\infty$ , then  $u$  is  $K$ -quasilinearly subharmonic (resp.  $K$ -quasilinearly subharmonic n.s.).
- (5) If  $u : D \rightarrow [-\infty, +\infty)$  is  $K_1$ -quasilinearly subharmonic and  $v : D \rightarrow [-\infty, +\infty)$  is  $K_2$ -quasilinearly subharmonic, then  $\max\{u, v\}$  is  $K$ -quasilinearly subharmonic in  $D$  with  $K = \max\{K_1, K_2\}$ . Especially,  $u^+ := \max\{u, 0\}$  is  $K_1$ -quasilinearly subharmonic.
- (6) Let  $\mathcal{F}$  be a family of  $K$ -quasilinearly subharmonic (resp.  $K$ -quasilinearly subharmonic n.s.) functions in  $D$  and let  $w := \sup_{u \in \mathcal{F}} u$ . If  $w$  is Lebesgue measurable and  $w^+ \in \mathcal{L}_{\text{loc}}^1(D)$ , then  $w$  is  $K$ -quasilinearly subharmonic (resp.  $K$ -quasilinearly subharmonic n.s.).



- (7) If  $u : D \rightarrow [-\infty, +\infty)$  is quasilinearly subharmonic n.s., then either  $u \equiv -\infty$  or  $u$  is finite almost everywhere, and  $u \in \mathcal{L}_{\text{loc}}^1(D)$ .

We leave the simple statements (1)–(6) to the reader and note only that a proof of (7) is completely similar to the proof of Lemma 1.

**Remark 2.** If  $u : D \rightarrow [-\infty, \infty)$  is strictly negative, finite and constant, then  $u$  is nearly subharmonic but, for every  $K > 1$ ,  $u$  is not  $K$ -nearly subharmonic n.s. Thus, the analog of statement (1) does not hold for functions which are quasilinearly subharmonic in narrow sense.

**Remark 3.** Related to statement (2) above, it is easy to see that, if  $u : D \rightarrow [-\infty, +\infty)$  is  $K$ -quasilinearly subharmonic, then  $\lambda u + C$  is  $K$ -quasilinearly subharmonic for all  $\lambda \geq 0$  and  $C \geq 0$ .

The following example shows that the sum of two quasilinearly subharmonic functions can be not quasilinearly subharmonic.

**Example 5.** The function  $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,

$$u(x, y) := \begin{cases} 3, & \text{when } x = 0, \\ 1, & \text{when } x \neq 0, \end{cases}$$

is 3-quasilinearly subharmonic. The constant function  $v : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,

$$v(x, y) \equiv -2,$$

is harmonic. Then we have

$$(u + v)(x, y) := \begin{cases} 1, & \text{when } x = 0 \\ -1, & \text{when } x \neq 0 \end{cases}$$

and

$$(u + v)_M = \max\{u + v, -M\} + M = (u + v + M)^+$$

for every  $M \geq 0$ . In particular for  $M = 1$  we obtain

$$(u + v)_1(x, y) := \begin{cases} 2, & \text{when } x = 0 \\ 0, & \text{when } x \neq 0. \end{cases}$$

Since  $(u + v)_1(0, 0) > 1$  and the double integral  $\iint_B (u + v)_1(x, y) dx dy$  is zero for every ball  $B \subset \mathbb{R}^2$ , the function  $(u + v)_1$  is not quasilinearly subharmonic. Hence  $(u + v)$  is also not quasilinearly subharmonic.

**Remark 4.** It is easy to see that the analog of statement (7) from Proposition 3 does not hold for quasilinearly subharmonic functions. A counterexample is the function  $u : \mathbb{R}^2 \rightarrow [-\infty, +\infty)$ ,

$$u(x, y) := \begin{cases} -\infty, & \text{when } y \leq 0, \\ 1, & \text{when } y > 0, \end{cases}$$

which is 2-quasilinearly subharmonic, but surely not quasilinearly subharmonic n.s.

#### 4. CHARACTERIZATION OF HARMONIC FUNCTIONS VIA QUASILINEARLY SUBHARMONIC FUNCTIONS

A subharmonic function  $u : \Omega \rightarrow [-\infty, \infty)$  defined on an open  $\Omega \subseteq \mathbb{R}^N$  is harmonic if and only if the function  $-u$  is also subharmonic, [14], p. 54. In this section we show that this remains true if one uses quasilinearly subharmonic in the narrow sense functions instead of subharmonic functions.

**Proposition 4.** *Let  $D$  be a domain in  $\mathbb{R}^N$ ,  $N \geq 2$ . Then the following statements are equivalent for every  $u : D \rightarrow [-\infty, \infty)$ .*

- (1) *The function  $u$  is harmonic.*
- (2) *There is  $K \geq 1$  such that the functions  $u$  and  $-u$  are  $K$ -quasilinearly subharmonic n.s.*

*Proof.* The implication (1)  $\Rightarrow$  (2) is trivial. Suppose statement (2) holds. Since  $u$  and  $-u$  are  $K$ -quasilinearly subharmonic n. s., we have

$$\frac{K}{\nu_N r^N} \int_{B^N(x,r)} u(y) dm_N(y) \leq u(x) \leq \frac{K}{\nu_N r^N} \int_{B^N(x,r)} u(y) dm_N(y)$$

and, consequently,

$$(6) \quad u(x) = \frac{K}{\nu_N r^N} \int_{B^N(x,r)} u(y) dm_N(y)$$

holds whenever  $\overline{B^N(x,r)} \subset D$ . Using statement (7) from Proposition 3 we see that  $u \in \mathfrak{L}_{loc}^1(D)$ . It follows from (6) for all  $x, z \in D$  and sufficiently small  $r > 0$  that

$$(7) \quad |u(x) - u(z)| \leq \frac{K}{\nu_N r^N} \int_{B^N(x,r) \triangle B^N(z,r)} |u(y)| dm_N(y),$$

where  $B^N(x,r) \triangle B^N(z,r)$  is the symmetric difference of the balls  $B^N(x,r)$  and  $B^N(z,r)$ . Since

$$\lim_{x \rightarrow z} m_N(B^N(x,r) \triangle B^N(z,r)) = 0,$$

the absolute continuity of the Lebesgue integral and the condition  $u \in \mathfrak{L}_{loc}^1(D)$  imply that  $f$  is continuous on  $D$ . Let  $x \in D$  and  $u(x) \neq 0$ . Equality (6) and continuity of  $u$  at the point  $x$  imply that  $K = 1$ . Every continuous function  $u$  satisfying (6) with  $K = 1$  for all  $B^N(x,r) \subset D$  is harmonic. Statement (1) follows.  $\square$

**Corollary 1.** *Let  $D$  be a domain in  $\mathbb{R}^N$ ,  $N \geq 2$ . Then a function  $u : D \rightarrow [-\infty, \infty)$  is harmonic if and only if the functions  $u$  and  $-u$  are nearly subharmonic.*

**Lemma 2.** *Let  $D$  be a domain in  $\mathbb{R}^N$ ,  $N \geq 2$ , let  $u : D \rightarrow [-\infty, \infty)$  be  $K_1$ -quasilinearly subharmonic n.s. and let  $-u$  be  $K_2$ -quasilinearly subharmonic n.s. If there is a point  $y_0 \in D$  such that  $u(y_0) > 0$ , then the inequality  $K_1 \geq K_2$  holds.*

*Proof.* Let  $y_0 \in D$  and  $u(y_0) > 0$ . Then for sufficiently small  $r > 0$  we have the double inequality

$$(8) \quad \frac{K_2}{\nu_N r^N} \int_{B^N(y_0, r)} u(y) dm_N(y) \leq u(y_0) \leq \frac{K_1}{\nu_N r^N} \int_{B^N(y_0, r)} u(y) dm_N(y),$$

thus

$$(9) \quad \frac{K_2}{\nu_N r^N} \int_{B^N(y_0, r)} u(y) dm_N(y) \leq \frac{K_1}{\nu_N r^N} \int_{B^N(y_0, r)} u(y) dm_N(y).$$

Inequality (8),  $u^+ \in \mathfrak{L}_{loc}^1(D)$  and  $u(y_0) > 0$  imply that

$$0 < \int_{B^N(y_0, r)} u(y) dm_N(y) < +\infty.$$

Now  $K_2 \leq K_1$  follows from (9).  $\square$

Using this lemma and Proposition 2 we obtain the following

**Proposition 5.** *Let  $D$  be a domain in  $\mathbb{R}^N$ ,  $N \geq 2$ . Let  $u : D \rightarrow [-\infty, \infty)$  be a function such that there are  $x_1, x_2 \in D$  satisfying the double inequality*

$$(10) \quad u(x_1) > 0 > u(x_2).$$

*Then the function  $u$  is harmonic if and only if the functions  $u$  and  $-u$  are quasilinearly subharmonic n.s.*

*Proof.* It suffices to show that  $u$  is harmonic if  $u$  and  $-u$  are quasilinearly subharmonic n.s. Let  $u$  be  $K_1$ -quasilinearly subharmonic n.s. and  $-u$  be  $K_2$ -quasilinearly subharmonic n.s. Then double inequality (10) and Lemma 2 imply the equality  $K_1 = K_2$ . Now the harmonicity of  $u$  follows from Proposition 4.  $\square$

The following example shows that there is  $u : D \rightarrow (0, \infty)$  such that  $u$  and  $(-u)$  are quasilinearly subharmonic n.s. but  $u$  is not harmonic.

**Example 6.** *Let  $D = \mathbb{R}^n$  and*

$$u(x) := \begin{cases} 2, & \text{when } x = 0, \\ 1, & \text{when } x \neq 0. \end{cases}$$

*Then  $u$  is 2-quasilinearly subharmonic n.s. and  $(-u)$  is 1-quasilinearly subharmonic n.s., but  $u$  is discontinuous at zero.*

**Remark 5.** *The above functions  $u$  and  $(-u)$  are both 2-quasilinearly subharmonic. Thus Proposition 4 becomes false if we replace the quasilinearly subharmonicity n.s. by quasilinearly subharmonicity.*

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